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A CRITICAL ANALYSIS ON VARIAIONAL INEQUALITIES

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ABSTRACT

It is well known that the minimum of a linear functional on a convex set in a Hilbert space can be characterized by a variational inequality. The same is proved for a class of differentiable and non-differentiable nonlinear functionals. Representation theorems are proved for nonlinear problems in a Hilbert space. The Riesz-Frechet theorem and the Lax-Milgram lemma can be deduced from these theorems. Techniques based on the contraction mapping theorem are used to prove the existence of a unique solution to a new class of nonlinear variational inequalities. It has been shown that the linearization of the variational inequalities is useful for the regularization approximation. The equivalence of variational and weak formulations of nonlinear boundary value problems is proved. A finite element approximation to the solution of the weak problem in a finite dimensional subspace of the original Hilbert space is defined. Using the concept of pseudo projection, an inequality bounding the error in this approximation over all functions of the subspace is derived. In the study of variational inequalities, it is necessary that in addition to the usual finite dimensional subspace of the Hilbert space, we construct finite dimensional convex subset of the Hilbert space. We note that this finite dimensional convex subset is not necessarily contained in the original convex subset of the original Hilbert space. New inequalities bounding the error in these approximations over the functions of the finite dimensional convex subset are derived for the nonlinear variational inequalities

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INTRODUCTION

The Riesz-Frechet representation theorem implies the existence of continuous linear function als along with a unique representation for each of them by an inner product in a Hilbert space. The fact that the minimum of a certain quadratic form does exist, which arises in the proof of the Riesz-Fr´echet theorem, is fundamental in variational problems.

To be more precise, let H be a real Hilbert space with its dual space H'. If a(u,v) is a continuous bilinear form and F is a continuous linear functional on H, then we consider a functional I[v], defined by

$$I[v] \equiv a\langle v, v\rangle - 2F(v), \qquad \text{for all } v \in H. \tag{0.1}$$

Lax and Millgram [3] considered the problem of finding u ∈ H such that

$$a\langle u, v\rangle = F(v), \qquad \text{for all } v \in H. \tag{0.2}$$

This problem is a natural extension (linear to bilinear), of the Riesz - Frechet theorem. It turns out for a symmetric bilinear form a(u,v) , that equations (0.2) are obtained by imposing on the functional I[v] the necessary conditions for obtaining a minimum on H. It has been shown [34] that the variational, (0.1), and weak, (0.2), formulations of linear elliptic boundary value problems are equivalent. Mikhlin [24] also proves that the minimum of a quadratic form on H

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can be characterized by the weak formulation. This formulation enables estimates to be obtained for the error in Finite element approximations to the solutions of linear elliptic boundary value problems derived using the Galerkin technique. Naturally a question arises as to whether or not the equivalence of the variational and weak solutions holds on a convex set in a Hilbert space. The fact this is so has been shown by Stampacchia [34], who characterized the minimum of the functional I[v] on a convex subset of H by a variational inequality. The convex subset here usually consists of admissible func tions, which in addition satisfy extra auxiliary conditions. Closely related to the variational inequalities are the unilateral problems [17]. These problems mainly arise in mechanics and physics, see [17, 15], where the material structure under consideration is submitted to unilat eral constraints. In the absence of the constraints, we have the variational equations.

In 1960, Minty [25] and Zarantonello [48] independently introduced the notion of a monotone operator. Thus it is possible to consider not merely bilinear forms, but also forms linear in only one of the two arguments. Making use of monotone operator theory, Browder [7] considers variational inequalities of the type (5.27) in Banach spaces, which are more general than and include the variational inequalities studied by Lions and Stampacchia [21], as a special case. These inequalities allow us to consider some nonlinear elliptic boundary value problems. Sibony [32, 33] has proved that the minimum of the functional I[v], when a(u,v) is a differentiable nonlinear functional, can be characterized by the variational inequalities introduced by Browder.

The variational inequality approach consists in dealing with such inequalities without assum ing a priori that the monotone operator involved is the Fr´echet derivative of a differentiable nonlinear functional. Brezis and Stampacchia [6], and Brezis [5] used this approch to study the regularity of the solution of the original problem. Recently Falk [16], and Mosco and Strang [28] have derived the error estimates for the approximation of a class of variational inequalities studied by Lions and Stampacchia.

We have noted that all these cases are extensions of the Riesz-Frechet representation theorem in various directions for the study of a wide class of linear and nonlinear boundary value problems. In all the previous investigations, the emphasis has been mainly on the form a(u,v) whilst F, as in (0.1), remains linear.

We consider the problem when F is a nonlinear continuous functional and a(u,v) is a bi-linear form. We study this problem in this thesis and explore some cases analogous to the Riesz-Fr´echet representation theorem and various generalizations. We show that a class of nonlinear boundary value problems can be studies by this technique. In every case, we have shown the relationship between our and the corresponding previous known results. In some cases, the method considered by us simplify the proofs of the previous known problems.

We now summarize some of these new results and the ideas used.

Chapter 1 begins with some preliminary definitions. We prove that for a differentiable non linear functional I[v], variational and weak formulations in a Hilbert space are equivalent. The minimum of both differentiable and non-differentiable nonlinear functionals on a convex subset of a Hilbert space is shown to be characterized by a class of nonlinear variational inequalities.

Chapter 2 contains the representation theorems for a class of nonlinear operators in a Hilbert space, which include the representation theorems of Riesz and Frechet, and Lax and Milgram as special cases. An iterative scheme is given which proves the existence of the representation theorems.

We consider a new class of nonlinear variational inequalities in chapter 3. The technique of Lions and Stampacchia has been used to prove the existence of a unique solution of these variational inequalities. Linearization of these variational inequalities is used to give a method of approximation.

The applications of the abstract theory developed in the previous chapters to nonlinear ellip tic boundary value problems are discussed in chapter 4.

In chapter 5, we derive the error bounds for the approximation of nonlinear boundary value problems via the weak formulation using the concept of pseudo projection. The inequalities bounding the error for the approximation of a class of nonlinear variational inequalities have been obtained. A simplified proof of a problem considered by Browder is given.

Formulation of Variational Inequalities

In this chapter, we prove that the minimum of differentiable and non-differentiable nonlinear functionals can be characterized by variational equations or inequalities over the whole Hilbert space or over a convex subset in the Hilbert space. We show that the formulations considered by us are more general and include as special cases all the corresponding previous formulations.

Let H be a real Hilbert space with its dual H', whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. The paring between H' and H is denoted by $\langle\cdot,\cdot\rangle$.

We now give some definitions, which will be used later on.

Definition 1.1 A function $F : H \to F$ (where $F = \mathbb{R}$ or $\mathbb{C}$), is a bounded linear functional, if for all $u, v \in H$,

(i). $F(\mu u + \lambda v) = \mu F(u) + \lambda F(v), \qquad$ for all $\mu, \lambda \in F$,

(ii). there exists a constant $\alpha > 0$ such that

$$|F(v)| \leq \alpha \| v \|, \qquad \text{for all } v \in H.$$

Note that continuity and boundedness are equivalent. A functional, which is not linear, is called nonlinear.

Definition 1.2 Let u, v be two different elements of H. The set of all elements $(1-t)u + tv$, where t assumes all values over the field F is called the straight line through u and v. If $t \in [0,1]$, we obtain the line segment between u and v. Moreover, a subset M of H is said to be convex, if $(1-t)u + tv \in M$, whenever $u, v \in M$ and for all $t \in [0,1]$.

From now onwards we denote by M, a convex subset, unless otherwise specified. A functional F is said to be convex on a convex subset M of H, if, for given u,v ∈ M, $0 \leq t \leq 1$,

$$F(tu + (1-t)v) \leq tF(u) + (1-t)F(v)$$

holds. F is a concave functional, if and only if, −F is convex.

Definition 1.3 A subset M of H is called a subspace, if $\mu u + \lambda v \in M$, whenever $u, v \in M$ and μ, λ are scalars. It is said to be closed, if $u_n \in M$, $u \in H$ such that $u_n \to u$ as $n \to \infty$, i.e., $\lim_{n\to\infty} \| u_n - u \| = 0$, implies that $u \in M$.

Definition 1.4 A bilinear form (functional) on H is a mapping $a : H \times H \to F$ satisfying;

(i). for each fixed element v in H, the mapping $u \to a(u,v)$ is a linear functional on H.

(ii). for each fixed element u in H, the mapping $v \to a(u,v)$ is a linear functional on H.

In case H is a complex Hilbert space, the condition (ii) is replaced by

(ii).* for each fixed u in H, the mapping $v \to a(u,v)$ is a antilinear functional on H, i.e.,

$$a(u, \alpha v) = \bar{\alpha} a(u,v), \qquad \text{for all } \alpha \in F.$$

Further, a bilinear form a(u,v) is said to be symmetric form, if

a(u,v) = a(v,u), for all u,v ∈ H.

Definition 1.5 A bilinear form a(u,v) on H is said to be positive definite (coercive [21], H-elliptic [41]), if there exists a constant ρ > 0 such that

$$a(u,v) \geq \rho \| v \|^2, \qquad \text{for all } v \in H,$$

and positive (non-negative), if

$$a(u,v) \geq 0, \qquad \text{for all } v \in H,$$

which is weaker assumption than positive definiteness. Note that positive definiteness implies positively but not conversely.

A bilinear form $a(u,v)$ is said to be continuous (bounded), if there exist a constant $\mu > 0$ such that

$$|a(u,v)| \leq \mu \parallel u \parallel \parallel v \parallel, \qquad \text{for all } u,v \in H.$$

The Cauchy-Schwartz inequality holds for the bilinear form $a(u,v)$ and is given by

$$|a(u,v)|^2 \leq a(u,u)a(v,v), \qquad \text{for all } u,v \in H.$$

Lemma 1.1 A bilinear form is continuous with respect to the norm convergence.

Proof. Let $u_n \to u$ and $v_n \to v$. Then these sequences are bounded. Let β be their bound, and $\parallel u_n \parallel \leq \beta$.

Now

$$\begin{aligned} |a(u_n,v_n) - a(u,v)| &= |a(u_n,v_n) - a(u_n,v) + a(u_n,v) - a(u,v)| \\ &\leq |a(u_n,v_n - v)| + |a(u_n - u, v)| \\ &\leq \mu\beta \parallel v_n - v \parallel + \mu \parallel u_n - u \parallel \parallel v \parallel . \end{aligned}$$

But $\parallel u_n - u \parallel \to 0$ and $\parallel v_n - v \parallel \to 0$ as $n \to \infty$, and therefore

$$|a(u_n,v_n) - a(u,v)| \to 0,$$

i.e., $a(u_n,v_n) \to a(u,v).$

Definition 1.6 A functional F : H → Y, where H and Y are normed linear spaces, is Fr´echet differentiable at u ∈ H, if there is an element F(u) ∈ L[H,Y], the space of all linear continuous functionals from H into Y, such that

$$F(u+v) - F(u) = F'(u)v + \epsilon(v), \qquad \text{for all } v \in H,$$

where $\frac{\|\epsilon(v)\|}{\|v\|} \to 0$ as $v \to 0$. It follows that, if $F'(u)$ exists, then

$$\lim_{t\to 0} \frac{F(u+tv) - F(u)}{t} = \langle F'(u), v\rangle, \qquad \text{for all } v \in H.$$

The operator $F' : H \to L[H,Y]$, which assigns $F'(u)$ to u called the Fréchet derivative of F and $\langle F'(u), v\rangle$ is called the Fréchet differential of the functional F in the direction v.

Since we are only interested in the case $Y = \mathbb{R}$ with H, a real Hilbert space, then it is clear [10] that $L[H,Y] = H'$, the dual of H. Thus from the definition of Fréchet derivative, it follows that $F'(u)$ is an element of the space H'.

Moreover, $F'(u)v$ being a continuous linear functional in $v \in H$ can be uniquely represented as a paring between H' and H, that is $F'(u)v = \langle F'(u), v\rangle$. Thus we have

$$\lim_{t\to 0} \frac{F(u+tv)-F(u)}{t} = \langle F'(u), v\rangle, \qquad \text{for all } u, v \in H.$$

We will always use this form of representation for the Fréchet differentiable. The Fréchet derivative $F'(u)$ means that the derivative of the functional F is taken with respect to the argument u.

Remark 1.1 The Fr´echet derivative always has its domain in the same space as the original functional. With this interpretation F and F have the same range space. For more details see Tapia [37], Nashed [29] and Carrol [10].

Remark 1.2 For a linear functional F, we note that, for all $u, v \in H$;

$$\begin{aligned}\langle F'(u), v\rangle &= \lim_{t\to 0} \frac{F(u+tv)-F(u)}{t} \\ &= \lim_{t\to 0} \frac{F(u)+tF(v)-F(u)}{t} \qquad F \text{ is linear} \\ &= F(v) \\ &\equiv \langle F, v\rangle, \qquad \text{for all } F \in H'.\end{aligned}$$

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Thus we see that for a linear functional F, $\langle F'(u), v\rangle = \langle F, v\rangle$.

This fact will play a very important part in showing the relationship between our results and previous known results for the linear case.

From now onward, differentiable functionals mean Fréchet differentiable functionals unless otherwise specified.

Definition 1.7 The operator $T : M \to H'$ is called antimonotone, if

$$\langle Tu - Tv, u - v\rangle \le 0, \qquad \text{for all } u, v \in M,$$

or T is antimonotone if and only if $-T$ is monotone [10].

We note from Remark 1.2 that every linear functional is both monotone and antimonotone. We have the following relationship between the antimonotonicity of the Fréchet derivative of a differentiable functional and the functional itself. This result is essentially due to Vainberg [40], but our method of proof is different.

Lemma 1.2 Let F be a real-valued differentiable functional on M. Then F is concave, if and only if, its Fréchet derivative F' is antimonotone.

Proof. Let F be a concave functional, then for all $u, v \in M$ and $t \in [0,1]$, $tu + (1-t)v \in M$, and we have

$$\begin{aligned} F(v+t(u-v)) &\equiv F(tu+(1-t)v) \\ &\geq tF(u)+(1-t)F(v), \qquad \text{by definition.} \end{aligned}$$

Rearranging and dividing by t, we get

$$\frac{F(v+t(u-v))-F(v)}{t} \geq F(u)-F(v).$$

Let t → 0. Since F is differentiable, it follows that F (v),u−v ≥F(u)−F(v).
Similarly, for all u, v ∈ M, F(u),v –u ≥ F(v)−F(u).
By adding, we have F(v)−F(u),v −u≤0,
showing that F is antimonotone. Conversely suppose that the Frechet derivative F is antimonotone. Since F is a differentiable functional, so we have the following representation [40].

$$F(u)-F(v) = \int_0^1 \langle F'(v+s(u-v)), u-v\rangle \mathrm{d}s. \qquad (1.1)$$

Consider, for all u,v ∈ M and t ∈ [0,1],

$$\begin{aligned} F(v+t(u-v))-F(v) &= \int_0^1 t\langle F'(v+ts(u-v)), u-v\rangle \mathrm{d}s \\ &= \int_0^1 t\langle F'(v+ts(u-v))-F'(v+s(u-v)), u-v\rangle \mathrm{d}s \\ &\quad + \int_0^1 t\langle F'(v+s(u-v)), u-v\rangle \mathrm{d}s. \end{aligned}$$

Let $U = v+s(u-v)$ and $V = v+st(u-v)$, then $U-V = s(1-t)(u-v)$, and so, since $(1-t) \geq 0$ for all $t \in [0,1]$, it follows by the antimonotonicity of F' that

$$0 \leq \langle F'(V)-F'(U), U-V\rangle = s(1-t)\langle F'(V)-F'(U), u-v\rangle.$$

Thus we have

$$\begin{aligned} F(v+t(u-v))-F(v) &\geq \int_0^1 t\langle F'(v+s(u-v)), u-v\rangle \mathrm{d}s \\ &= t\{F(u)-F(v)\}, \quad \text{by } (5.51). \end{aligned}$$

That is

$$F(v+t(u-v)) \geq tF(u)+(1-t)F(v),$$

the required result.

If a(u,v) is a continuous bilinear form on H and F is a real-valued continuous functional, then we consider a functional I[v] defined by

$$I[v] = a(v,v) - 2F(v), \qquad \text{for all } v \in H. \qquad (1.2)$$

Many mathematical problems either arise or can be formulated in term of functional of this form. Here one seeks to minimize the functional I[v] defined by (5.52) over a whole space or a convex subset M of H bearing in mind whether the real-valued functional F is linear or not. We point out that the whole theory of variational methods can be based on the minimum of the functional I[v]. For the nonlinear form a(u,v) and linear functional F, it has been shown [33] that the minimum of I[v] on H (or M ⊂ H) can be characterized by a variational equation (or inequality). Furthermore, there arise two cases according as F is linear or nonlinear. The case F is linear has been studied by Stampacchia [34] and Temam [38]. We state their result without proof.

Theorem 1.1 If F ∈ H and a(u,v) is a positive, symmetric and continuous bilinear form on H, then the function u ∈ H minimizes I[v], defined by (5.52), if and only if

$$a(u,v) = \langle F, v\rangle \qquad \text{for all } v \in H. \qquad (1.3)$$

A result analogous to Theorem 1.1 is now established for the case when F is a nonlinear differentiable functional.

Theorem 1.2 Let a(u,v) be a positive, summetric and continuous bilinear form on H. If the Frechet derivative F of a nonlinear functional F defined on H exists and is antimonotone, then the function u ∈ H minimizes I[v] if and only if

$$a(u,v) = \langle F'(u), v\rangle \qquad \text{for all } v \in H. \qquad (1.4)$$

Proof. Let u minimize $I[v]$, then for all $\lambda \in \mathbb{R}$ and $w \in H$, $v = u + \lambda w \in H$,

$$I[u] \le I[u + \lambda w].$$

Thus from (5.52), it follows that

$$\begin{aligned} a(u,u) - 2F(u) &\le a(u+\lambda w, u+\lambda w) - 2F(u+\lambda w) \\ &= a(u,u) + \lambda 2a(u,w) + \lambda^2 a(w,w) - 2F(u+\lambda w), \end{aligned}$$

and so

$$a(u,w) \ge \frac{F(u+\lambda w) - F(u)}{\lambda} - \frac{\lambda}{2} a(w,w). \qquad (1.5)$$

Let $\lambda \to 0$. Since F is differentiable, we have

$$a(u,w) \ge \langle F'(u), w\rangle \qquad \text{for all } w \in H.$$

For now replacing w by $-w \in H$, we have

$$a(u,w) \le \langle F'(u), w\rangle \qquad \text{for all } w \in H.$$

Thus we obtain

$$a(u,w) = \langle F'(u), w\rangle \qquad \text{for all } w \in H.$$

We shall use this device several times in this thesis.

For the converse, if $u \in H$ satisfies (5.57), then using the positivity of $a(u,v)$, we have

$$\begin{aligned} I[u] - I[v] &= a(u,u) - 2F(u) - a(v,v) - 2F(v) \\ &= a(u,u-v) - a(v,v-u) - 2F(u) + 2F(v) \\ &= 2a(u,u-v) - 2a(v-u,v-u) - 2F(u) + 2F(v) \\ &\le 2\langle F'(u), u-v\rangle + 2F(v) - 2F(u), \qquad \text{by (5.57).} \end{aligned}$$

Since F' is antimonotone, hence by Lemma 1.2, it is concave. Thus

$$a\langle F'(u), u-v\rangle + 2F(v) - 2F(u) \leq 0, \qquad \text{for all } u, v \in H,$$

so that

$$I[u] \leq I[v].$$

Note that for the case when F is a linear functional, it follows that the result of Theorem 1.2 is exactly that given in Temam [38]. In this case it is seen from (1.5) that

$$a(u,w) + \frac{\lambda}{2}a(w,w) \geq F(w), \qquad \text{for all } w \in H,$$

We again consider the following two cases;
(i). For the linear case, one can easily show that for each linear continuous functional F on H, the function $u \in H$ minimizes $I_1[v]$ if and only if

$$(u,v) = \langle F, v\rangle, \qquad \text{for all } v \in H.$$

Thus we have the variational character of the Riesz-Fréchet representation theorem, see Theorem 5.10.
For the nonlinear differential functional F, we have the following result.

Remark 1.3 If the Fréchet derivative F' of a nonlinear differential functional F exists and is antimonotone, then the function $u \in H$ minimizes $I_1[v]$ if and only if

$$(u,v) = \langle F'(u), v\rangle, \qquad \text{for all } v \in H,$$

This can be viewed as the variational formulation of the Riesz-Fréchet theorem for a class of differentiable nonlinear functionals.

For minimizing $I[v]$ over a convex set M in H instead of the entire space H, the variational equation turns into inequality. The case when F is linear, has been studied by Stampacchia [34] and Lions-Stampacchia [21], and is known as one of the first and most fundamental of the variational problems. A similar idea has been used by Fichera [17] in the study of unilateral constraint problems in elasticity. We state their result without proof.

Theorem 1.3 Let $a(u,v)$ be a positive definite, symmetric and continuous bilinear form and M a convex subset of H. If $F \in H'$, then the function $u \in M$ minimizes $I[v]$, defined by (5.52), if and only if

$$a(u, v-u) \geq \langle F, v-u\rangle, \qquad \text{for all } v \in M. \qquad (1.6)$$

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